MATH 2020 A Tutorial 5 29/10/2015

Topic:
- Volume Calculation
- Volume of n-dimensional ball.

Defn: (Volume of a region R in $\mathbb{R}^n$)

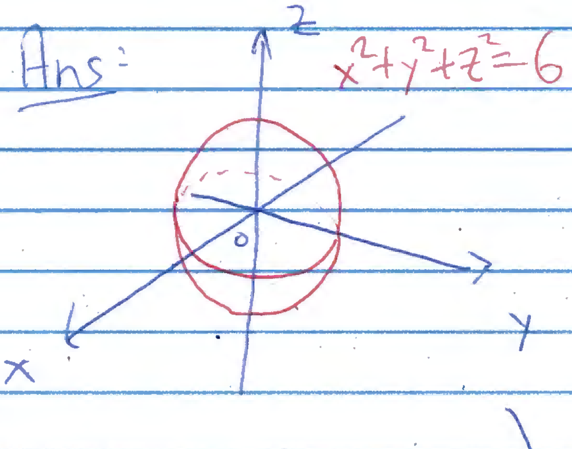
Let $R \subseteq \mathbb{R}^n$ be a region.
Then then volume of R in $\mathbb{R}^n$ is defined by

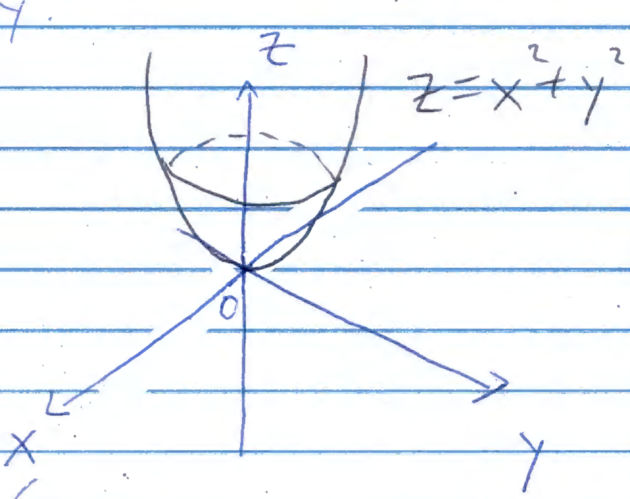
$$V_R = \iiint_R 1 \, dV$$

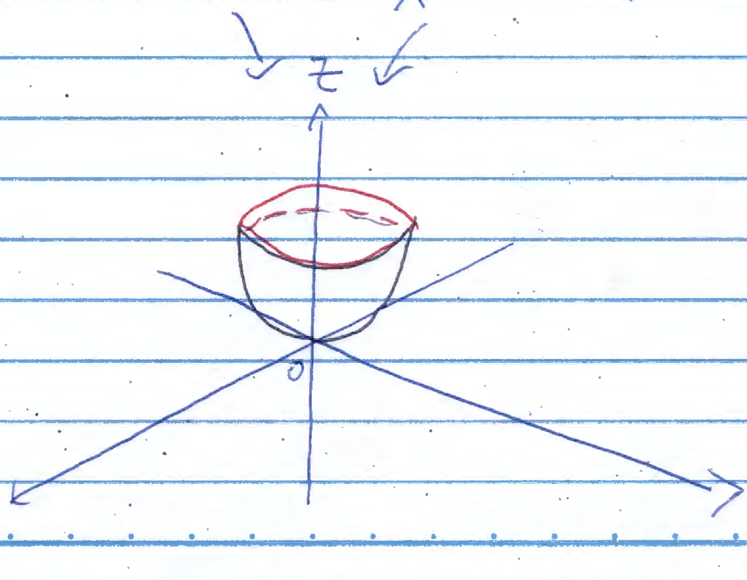
Volume Calculation:

Example: (1) Find the volume of the solid bounded by the sphere $x^2+y^2+z^2=6$ and the paraboloid $z=x^2+y^2$.

Ans:







Note that $\begin{cases} x^2+y^2+z^2=6 \\ z=x^2+y^2 \end{cases}$

$$\Rightarrow \quad z^2+z-6=0$$

$$\Rightarrow \quad (z+3)(z-2)=0$$

$$\Rightarrow \quad z=2 \quad \text{or } -3 \text{ (rejected as } z \geq 0\text{)}$$

$$\Rightarrow \quad x^2+y^2=2$$

$$\therefore \quad V_R = \iiint_R dV$$

$$= \iint_{\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\leq 2\}} \left[\sqrt{6-x^2-y^2} - (x^2+y^2)\right] dx\,dy$$

(Polar Coordinate)

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} \left[\sqrt{6-r^2} - r^2\right] r\,dr\,d\theta$$

$$= 2\pi\left(\frac{1}{2}\right)\int_0^{\sqrt{2}} \left(\sqrt{6-r^2}-r^2\right) dr^2$$

$$= \pi\left(-\int_0^{\sqrt{2}} \sqrt{6-r^2}\,d(6-r^2) - \int_0^{\sqrt{2}} r^2\,dr^2\right)$$

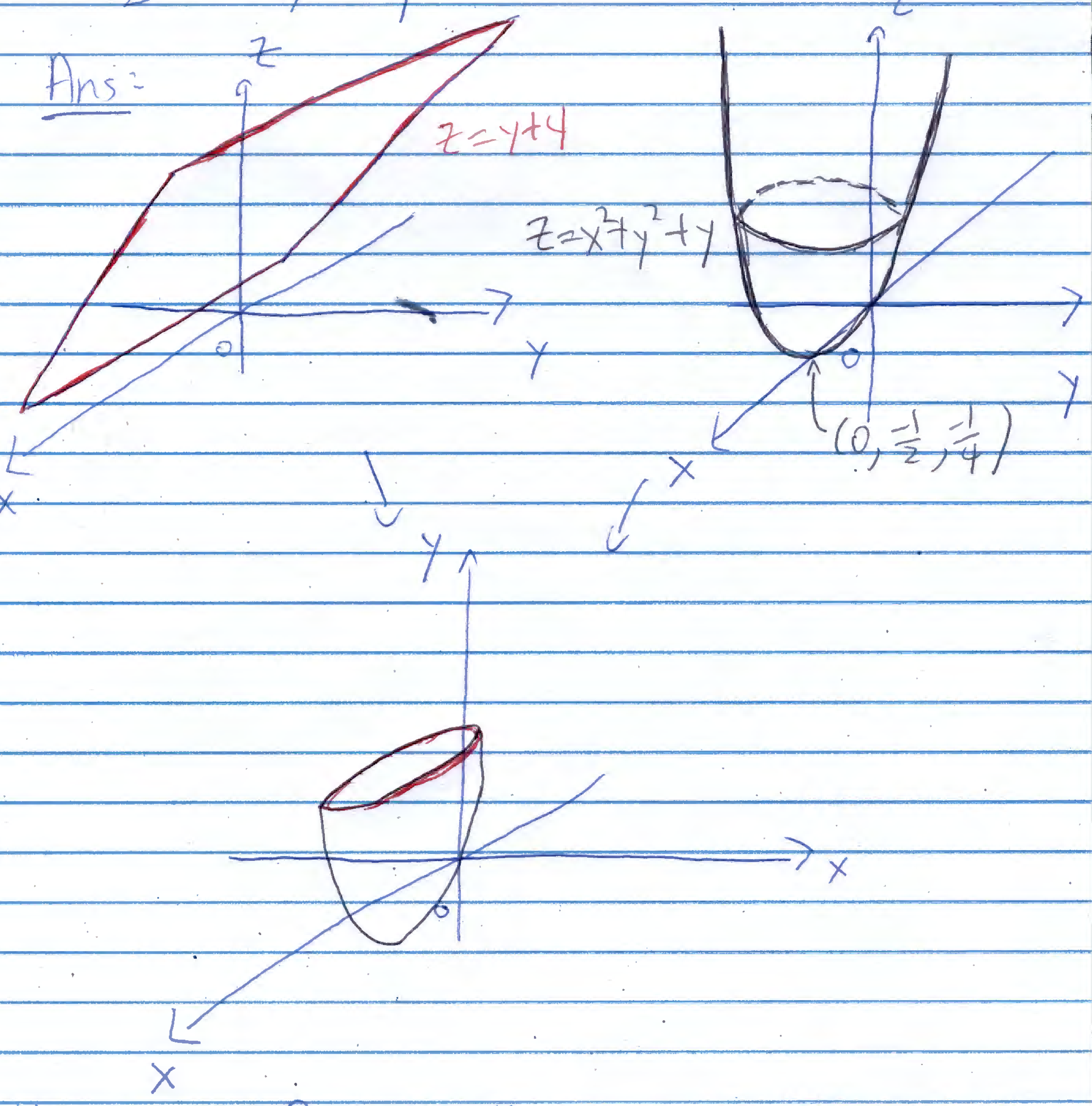
$$= \pi\left(-\left[\frac{2}{3}(6-r^2)^{\frac{3}{2}}\right]_0^{\sqrt{2}} - \left[\frac{(r^2)^2}{2}\right]_0^{\sqrt{2}}\right)$$

$$= \pi\left(\frac{2}{3}(6)^{\frac{3}{2}} - \frac{22}{3}\right)$$

//

(2) Find the volume of the solid bounded by the plane $z=y+4$ and the paraboloid $z=x^2+y^2+y$.

Ans:



Note that $\begin{cases} z=y+4 \\ z=x^2+y^2+y \end{cases} \Rightarrow x^2+y^2=4.$

$$\therefore V_R = \iiint_R 1\, dV$$

$$= \iint_{\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 4\}} \left[(y+4)-(x^2+x^2-y)\right] dx\,dy$$

Polar Coordinate

$$= \iint_{\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 4\}} (4-x^2-y^2)\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^2 (4-r^2)\,r\,dr\,d\theta$$

$$= 2\pi \int_0^2 (4r-r^3)\,dr$$

$$= 2\pi \left[2r^2 - \frac{r^4}{4}\right]_0^2$$

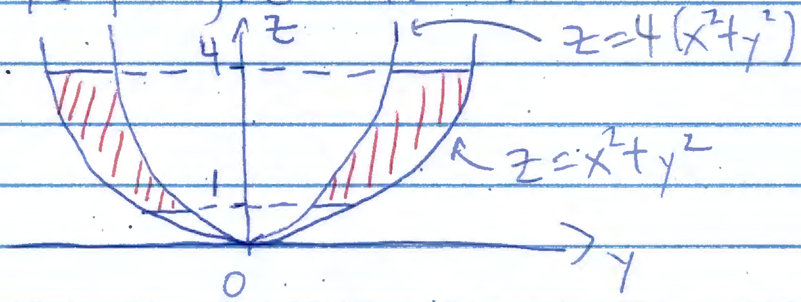
$$= 8\pi$$

(3) Using the transformation

$x=\frac{r}{t}\cos\theta,\ y=\frac{r}{t}\sin\theta,\ z=r^2,\ r\geq 0,\ t\geq 0,\ \theta\in[0,2\pi]$

to find the volume of the region bounded by the paraboloids $z=x^2+y^2$, $z=4(x^2+y^2)$ and between the plane $z=1$ and $z=4$.

Ans: In yz-plane, it looks like



Change of Variable Formula:

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_{R'} f(u,v,w)\left|\det\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

$\boxed{\text{Note that the boundary of R will be mapped to the boundary of R'}}$

① $z = x^2+y^2 \Rightarrow r^2 = (\frac{r}{t}\cos\theta)^2 + (\frac{r}{t}(\sin\theta)^2 \Rightarrow \frac{r^2}{t^2} = r^2 \Rightarrow t = 1$

② $z = 4(x^2+y^2) \Rightarrow r^2 = \frac{4r^2}{t^2} \Rightarrow t = 2$

③ $z = 1 \Rightarrow r^2 = 1 \Rightarrow r = 1$

④ $z = 4 \Rightarrow r^2 = 4 \Rightarrow r = 2$

$$\det \frac{\partial(x,y,z)}{\partial(r,t,\theta)} = \begin{vmatrix} x_r & y_r & z_r \\ x_t & y_t & z_t \\ x_\theta & y_\theta & z_\theta \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\cos\theta}{t} & \frac{\sin\theta}{t} & 2r \\ \frac{-r\cos\theta}{t^2} & \frac{-r\sin\theta}{t^2} & 0 \\ \frac{-r\sin\theta}{t} & \frac{r\cos\theta}{t} & 0 \end{vmatrix}$$

$$= -\frac{2r^3}{t^3}$$

$$\therefore V_R = \iiint_R 1 \, dV$$

$$= \int_0^{2\pi}\int_1^2\int_1^2 \frac{2r^3}{t^3}\, dr\, dt\, d\theta$$

$$= 2\pi(2)\left(\int_1^2 r^3 dr\right)\left(\int_1^2 \frac{1}{t^3} dt\right)$$

$$= \frac{45\pi}{8} \quad //$$

(4) Find the volume of the n-dimensional ball $B_n(R)$, where $B_n(R) = \left\{ (x_1, \cdots, x_n) \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^{n} x_i^2 \leq R^2 \right\}$

Ans: First, we prove the following proposition:

Prop: The volume is proportional to the nth power of the radius.

Pf: For $n=2$, $V_2(R) = \pi R^2 \propto R^2$.

Assume it is true for $n \in \mathbb{N}$.

i.e. $V_n(R) = kR^n$ for some $k \neq 0$.

Note that the hypothesis implies:

① $V_n(R) = kR^n$.

Put $R=1$, $V_n(1) = k$.

$\therefore V_n(R) = R^n V_n(1)$

② $V_n(R_0) = V_n(1) R_0^n$

$$\Rightarrow \frac{V_n(R_0)}{R^n} = \frac{V_n(1) R_0^n}{R^n}$$

$$\Rightarrow \frac{V_n(R_0)}{R^n} = V_n(1) \left(\frac{R_0}{R}\right)^n$$

$$\overset{\text{By ①}}{\Rightarrow} \frac{V_n(R_0)}{R^n} = V_n\left(\frac{R_0}{R}\right)$$

$$\Rightarrow V_n(R_0) = R^n V_n\left(\frac{R_0}{R}\right)$$

As a result,

$$V_{n+1}(R) \underset{\text{thm}}{\overset{\text{Fubini's}}{=\!=}} \int_{-R}^{R} \left(\iint\!\!\int_D dx_2 \cdots dx_n \right) dx_1,$$

where $D = \left\{ (x_2, \cdots, x_{n+1}) \in \mathbb{R}^n \,\middle|\, \sum_{i=2}^{n+1} x_i^2 \leq R^2 - x_1^2 \right\}$

is a n-dimensional ball with radius $\sqrt{R^2 - x_1^2}$

$$\left(\text{By } ②\right) = \int_{-R}^{R} V_n\left(\sqrt{R^2 - x_1^2}\right) dx_1$$

$$= R^n \int_{-R}^{R} V_n\left(\sqrt{1 - \left(\frac{x_1}{R}\right)^2}\right) dx_1$$

$$\left(\begin{array}{c}\text{Substitute} \\ t = \frac{x}{R}, \\ dx = R\,dt\end{array}\right) = R^{n+1} \int_{-1}^{1} V_n\left(\sqrt{1 - t^2}\right) dt$$

$$= R^{n+1} V_{n+1}(1)$$

$$\left(\begin{array}{l} V_{n+1}(R) = \int_{-R}^{R} V_n\left(\sqrt{R^2 - x_1^2}\right) dx_1 \\ \text{Put } R = 1, \\ \quad V_{n+1}(1) = \int_{-1}^{1} V_n\left(\sqrt{1 - x_1^2}\right) dx_1 \end{array}\right)$$

By Induction, we prove the prop.

Then, we will try to find a recursion formula for $V_n(R)$.

<u>Recursion formula for $V_n(R)$:</u>

$$V_n(R) \overset{\text{Fubini's}}{\underset{\text{Thm}}{=}} \iint_{D_1} \left(\iiint_{D_2} dx_3 \cdots dx_n \right) dx_1 dx_2,$$

where $D_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq R^2\}$

$D_2 = \left\{(x_3, \dots, x_n) \in \mathbb{R}^{n-2} \;\middle|\; \sum_{i=3}^{n-2} x_i^2 \leq R^2 - x_1^2 - x_2^2\right\}$

$$\left(\begin{array}{c}\text{Polar} \\ \text{Coordinate}\end{array}\right) = \iint_{D_1} V_{n-2}\left(\sqrt{R^2 - (x_1^2 + x_2^2)}\right) dx_1 dx_2$$

$$= \int_0^{2\pi} \int_0^R V_{n-2}\left(\sqrt{R^2 - r^2}\right) r\,dr\,d\theta$$

$$= 2\pi \int_0^R V_{n-2}(\sqrt{R^2-r^2})\, r\, dr$$

(By Prop.)

$$= 2\pi V_{n-2}(R) \int_0^R \frac{V_{n-2}(\sqrt{R^2-r^2})}{V_{n-2}(R)}\, r\, dr$$

$$= 2\pi V_{n-2}(R) \int_0^R \frac{(\sqrt{R^2-r^2})^{n-2}}{R^{n-2}} \frac{V_{n-2}(1)}{V_{n-2}(1)}\, r\, dr$$

$$= 2\pi V_{n-2}(R) \int_0^R \left[1-\left(\frac{r}{R}\right)^2\right]^{\frac{n-2}{2}} r\, dr$$

$$= 2\pi V_{n-2}(R)\left(\frac{-R^2}{2}\right) \int_0^R \left[1-\left(\frac{r}{R}\right)^2\right]^{\frac{n-2}{2}} d\left(1-\left(\frac{r}{R}\right)^2\right)$$

$$= 2\pi V_{n-2}(R)\left(\frac{-R^2}{2}\right)\left[\frac{1}{\frac{n-2}{2}+1}\left[1-\left(\frac{r}{R}\right)^2\right]^{\frac{n-2}{2}+1}\right]_0^R$$

$$= 2\pi V_{n-2}(R)\left(\frac{-R^2}{2}\right)\left(\frac{-1}{\frac{n-2}{2}+1}\right)$$

$$= \frac{2\pi R^2}{n} V_{n-2}(R).$$

$$\therefore V_n(R) = \frac{2\pi R^2}{n} V_{n-2}(R)$$

As a result, for $n=2k$ even, we have

$$V_{2k}(R) = \frac{2\pi R^2}{(2k)} V_{2k-2}(R)$$

$$= \frac{(2\pi R^2)^2}{(2k)(2k-2)} V_{2k-4}(R)$$

$$= \cdots\cdots$$

$$= \frac{(2\pi R^2)^{k-1}}{(2k)(2k-2)\cdots(4)} V_2(R)$$

$$= \frac{2^{k-1} \pi^{k-1} R^{2k-2}}{2^{k-1}(k!)} (\pi R^2)$$

$$= \frac{\pi^k R^{2k}}{(k!)}$$

Similarly, for $n=2k+1$ odd, we have

$$V_{2k+1}(R) = \frac{2\pi R^2}{(2k+1)} V_{2k-1}(R)$$

$$= \frac{(2\pi R^2)^2}{(2k+1)(2k-1)} V_{2k-3}(R)$$

$$= \cdots\cdots$$

$$= \frac{(2\pi R^2)^{k-1}}{(2k+1)(2k-1)\cdots(5)} V_3(R)$$

$$= \frac{2^{k-1} \pi^{k-1} R^{2k-2}}{(2k+1)(2k-1)\cdots(5)} \left(\frac{4}{3}\pi R^3\right)$$

$$= \frac{2^{k+1} \pi^k R^{2k+1}}{(2k+1)(2k-1)\cdots(5)}$$

$$= \frac{2(k!)\pi^k R^{2k+1}}{(2k+1)!}$$

$\boxed{///}$

Reference: Wikipedia, "Volume of an n-ball".